Ported Loudspeaker Frequency Response

An approach that even a mathematician can understand

Selwyn Hollis
Professor of Mathematics
Armstrong State University

1. Introduction

Initially this note’s only purpose was to put down in one place the complete system of differential equations that corresponds to the common model of ported (or “vented”/“bass-reflex”) loudspeaker enclosures driven by low-frequency, small-amplitude input. If that is documented elsewhere, it will come as a surprise to this author.

As our study developed, we realized that the determination of optimal enclosure parameters (for given driver parameters) could be accomplished in an efficient, straightforward way by numerically solving a system of nonlinear equations. An advantage of this approach is that it allows each of the usual three “alignments” (third-order quasi-Butterworth and fourth-order Butterworth/Chebyshev) to be handled in a unified way, leading to what essentially amounts to a continuous parameterization of all of these alignments. The method is implemented in publicly available web-based software [7] developed in Mathematica [8].

Acoustical engineers are apparently adept at analyzing systems such as those of interest here by means of analogous electrical circuits and using well-known principles of circuits to deduce the transfer functions of the acoustic system of interest. The apparent authoritative modern text on this and related topics is the book by Leach [1]. Our approach is to use instead the governing differential equations to derive the transfer functions.

All of this is related to works of Neville Thiele [2] and Richard Small [3] that were published in 1961 and 1973, respectively, and caused certain driver (i.e., speaker) parameters to become famously known as the Thiele-Small Parameters. Earlier work by Beranek [4], Novak [5], and others laid the foundations for the more well-known work of Thiele and then Small. For a historical timeline of the development of loudspeaker design during the mid-20th century, see [6].

It is not the purpose of this note to discuss any of the aspects of ported loudspeaker design beyond the basic mathematical model. The effects of amplifier/crossover impedances, enclosure shape/dimensions, flaring of port ends, construction materials, and enclosure stuffing have been addressed by numerous others, and we have nothing to add.

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2. List of parameters and constants

**Driver suspension characteristics**
- \( m_d \): mass (incl. air load)
- \( r_d \): mechanical damping coefficient
- \( k_d \): mechanical stiffness
- \( V_{as} \): box volume with equivalent stiffness
- \( \omega_s \): resonant angular frequency
- \( S_d \): effective surface area
- \( B/l \): magnetic flux density x voice-coil length
- \( R_e \): voice-coil DC resistance
- \( \omega_d \): resonant angular frequency
- \( Q_{ts} \): total quality factor

**Enclosure and port characteristics**
- \( V_{box} \): box volume
- \( A_p \): port cross-sectional area
- \( L_p \): port length
- \( m_p \): mass of air in port
- \( k_p \): box stiffness seen by the port
- \( \omega_p \): resonant angular frequency
- \( r_L \): coefficient of damping due to box losses
- \( Q_L \): box quality factor

**Constants**
- \( \rho \): mass density of air
- \( c \): speed of sound
- \( \rho c^2 \): adiabatic bulk modulus of air

**System parameters**
- \( \alpha = k_p/k_d \approx V_{as}/V_{box} \)
- \( h = \omega_p/\omega_d \)
- \( \omega_0 = \sqrt{\omega_p \omega_d} \)

3. The masses and forces involved

Let \( x_d \) and \( x_p \) denote the displacements of the driver diaphragm and port air mass, respectively, with the positive direction for both pointing out of the box. The resulting (small) change in the volume of the enclosure is then \( S_d x_d + A_p x_p \), which we will refer to as the *volume displacement*.

**The driver diaphragm and voice coil**

The mass of the diaphragm and voice coil, \( m_d \), is assumed to include all of its moving parts as well as its air load. The forces on it are as follows.

- **Acoustical**
  
  A small amplitude approximation to the difference in pressures on each side of the diaphragm is
  
  \[
  dp = -\frac{\rho c^2 V_{box}}{V^2} dV \approx -\frac{\rho c^2}{V_{box}} \Delta V = -\frac{\rho c^2}{V_{box}} (S_d x_d + A_p x_p). 
  \]

  The resulting force is that multiplied by \( S_d \):
  
  \[
  F_{ad} = -\frac{\rho c^2 S_d}{V_{box}} (S_d x_d + A_p x_p). 
  \]

- **Mechanical**
  
  These are the usual damped oscillator terms; \( k_d \) is the stiffness of the suspension, and \( r_d \) is the mechanical damping coefficient:
  
  \[
  F_{md} = -k_d x_d - r_d x_d'. 
  \]

  The parameter \( V_{as} \) is the volume of the box that provides the same stiffness/compliance as the suspension of the driver; i.e., \( k_d = \rho c^2 S_p^2 / V_{as} \). Thus
  
  \[
  F_{md} = -\frac{\rho c^2 S_p^2}{V_{as}} x_d - r_d x_d'. 
  \]
Electrical

For an input signal $E(t)$ volts, the resulting magnetic force is

$$F_{ed} = \frac{Bl}{Re} (E(t) - Blx'_d).$$

The air mass in the port

The mass of the air in the port is

$$m_p = \rho A_p \left( L_p + 1.46 \sqrt{A_p/\pi} \right),$$

the expression $L_p + 1.46 \sqrt{A_p/\pi}$ being the effective length of the port. The forces on this mass are as follows.

Acoustical

The difference in pressures on each side of the port is the same as in (1) above. The resulting force is that multiplied by $A_p$:

$$F_{ap} = - \frac{\rho c^2 A_p}{V_{box}} (S_d x_d + A_p x_p).$$

Enclosure energy loss

This is modeled as a force that is proportional to the velocity of the displaced volume:

$$F_{Lp} = -f_L (S_d x_d + A_p x_p').$$

4. The differential equations

Here are the differential equations that govern the displacements $x_d$ and $x_p$:

$$m_d x''_d = F_{md} + F_{ad} + F_{ed}
= - \frac{\rho c^2 S_d^2}{V_{as}} x_d - r_d x'_d - \frac{\rho c^2 S_d}{V_{box}} (S_d x_d + A_p x_p) + \frac{Bl}{Re} (E(t) - Blx'_d)$$

$$m_p x''_p = F_{ap} + F_{Lp}
= - \frac{\rho c^2 A_p}{V_{box}} (S_d x_d + A_p x_p) - r_L (S_d x'_d + A_p x'_p)$$

In terms of volume displacements $u_d = S_d x_d$ and $u_p = A_p x_p$, the equations become

$$m_d u''_d + \left( r_d + \frac{B^2 l^2}{Re} \right) u'_d + \frac{\rho c^2 S_d^2}{V_{as}} u_d + \frac{\rho c^2 S_d}{V_{box}} (u_d + u_p) = S_d \frac{Bl}{Re} E(t)$$

$$m_p u''_p + A_p r_L (u'_d + u'_p) + \frac{\rho c^2 A_p^2}{V_{box}} (u_d + u_p) = 0$$

We’ll tidy these up some by rewriting them in terms of

$$R_{td} = r_d + \frac{B^2 l^2}{Re}, \quad k_d = \frac{\rho c^2 S_d^2}{V_{as}}/V_{box}, \quad \alpha = \frac{V_{as}}{V_{box}}, \quad k_p = \frac{\rho c^2 A_p^2}{V_{box}}, \quad \varphi(t) = \frac{S_d Bl}{m_d Re} E(t).$$

(11)
A little rearrangement now gives

\[ m_d u_d'' + R_{ld} u_d + (1 + \alpha) k_d u_d = m_d \varphi(t) - \alpha k_d u_p \]
\[ m_p u_p'' + A_p R_L u_p' + k_p u_p = -k_p u_d - A_p R_L u_d \]  

(12)

Next we divide the equations by \( m_d \) and \( m_p \), respectively, and set

\[ \omega_d = \sqrt{k_d/m_d}, \quad \omega_p = \sqrt{k_p/m_p}, \quad Q_{ts} = \frac{\omega_d}{R_{ld}}, \quad Q_L = \frac{\omega_p}{A_p R_L}. \]  

(13)

Then (4) becomes

\[ u_d'' + \frac{\omega_d}{Q_{ts}} u_d' + (1 + \alpha) \omega_d^2 u_d = \varphi(t) - \alpha \omega_d^2 u_p \]
\[ u_p'' + \frac{\omega_p}{Q_L} u_p' + \omega_p^2 u_p = -\omega_p^2 u_d - \frac{\omega_p}{Q_L} u_d' \]  

(14)

Thus the system is determined by the resonant angular frequencies \( \omega_d \) and \( \omega_p \), the dimensionless parameters \( \alpha, Q_{ts} \) and \( Q_L \), and the forcing function \( \varphi(t) \). So for the purpose of plotting the frequency response the only driver parameters necessary are \( \omega_d, V_{as}, \) and \( Q_{ts} \). Those, along with box parameters \( A_p, L_p, V_{box}, \) and \( Q_L \), are sufficient to determine all the coefficients in (14).

The quality factor \( Q_L \) is the wild card here. Its value is not easily knowable, but it normally falls between 3 and 20 (according to Leach [1]). Usually one wants \( Q_L \) to be as large as practicable. Air leaks in the enclosure are a typical cause of a low \( Q_L \).

5. Derivation of the transfer functions

We’ll now define operators

\[ D = \frac{d}{dt}, \quad L_1 = \mathcal{P}_1(D), \quad L_2 = \mathcal{P}_2(D), \quad L_3 = \mathcal{P}_3(D) \]

based on the polynomials

\[ \mathcal{P}_1(s) = s^2 + \frac{\omega_d}{Q_{ts}} s + (1 + \alpha) \omega_d^2 \]
\[ \mathcal{P}_2(s) = s^2 + \frac{\omega_p}{Q_L} s + \omega_p^2 \]
\[ \mathcal{P}_3(s) = \frac{\omega_p}{Q_L} s + \omega_p^2 = \mathcal{P}_2(s) - s^2 \]  

(15)

(We remark that the operators \( L_1, L_2, L_3 \) commute; i.e., \( L_i L_j = L_j L_i \).)

The differential equations can now be written as

\[ L_1 u_d = \varphi(t) - \alpha \omega_d^2 u_p \]
\[ L_2 u_p = -L_3 u_d \]  

(16)

Next we decouple these equations. Applying \( L_2 \) to the first equation in (16) gives

\[ L_2 L_1 u_d = L_2 \varphi(t) - \alpha \omega_d^2 L_2 u_p \]
\[ = L_2 \varphi(t) + \alpha \omega_d^2 L_3 u_d \]

Thus

\[ (L_1 L_2 - \alpha \omega_d^2 L_3) u_d = L_2 \varphi(t) \].
This implies that the transfer function for the mapping $\varphi(t) \mapsto u_d$ is

$$G_d(s) = \frac{P_2(s)}{P_1(s) P_2(s) - \alpha \omega_d^2 P_3(s)}.$$  \hspace{1cm} (17)

Applying $L_1$ to the second equation in (16) gives

$$L_2 L_1 u_p = -L_1 L_3 u_d$$

$$= -L_3 (\varphi(t) - \alpha \omega_d^2 u_p).$$

Thus

$$(L_1 L_2 - \alpha \omega_d^2 L_3) u_p = -L_3 \varphi(t).$$

This implies that the transfer function for $\varphi(t) \mapsto u_p$ is

$$G_p(s) = \frac{-P_3(s)}{P_1(s) P_2(s) - \alpha \omega_d^2 P_3(s)}.$$ \hspace{1cm} (18)

It also follows that

$$(L_1 L_2 - \alpha \omega_d^2 L_3)(u_d + u_p) = (L_2 - L_3) \varphi(t)$$

$$= D^2 \varphi(t)$$

So the transfer function for the mapping $\varphi(t) \mapsto u_d + u_p$ is

$$G_c(s) = G_d(s) + G_p(s) = \frac{s^2}{P_1(s) P_2(s) - \alpha \omega_d^2 P_3(s)}.$$ \hspace{1cm} (19)

Consequently, with $E(t) = e_g e^{i \omega t}$ and $A_0 = \frac{S_b B_t}{m_s R_s} e_g$, we have $\varphi(t) = A_0 e^{i \omega t}$ and thus the following responses:

$$u_d(t) = G_d(i \omega) A_0 e^{i \omega t}$$

$$u_p(t) = G_p(i \omega) A_0 e^{i \omega t}$$

$$u_d(t) + u_p(t) = G_c(i \omega) A_0 e^{i \omega t}$$

with amplitudes $| G_d(i \omega) A_0 |$, $| G_p(i \omega) A_0 |$, $| G_c(i \omega) A_0 |$, respectively.

6. Equivalence with the standard formulas

In this section we will use Mathematica to assist with the algebra.

In addition to the stiffness ratio $\alpha = k_p / k_d = V_{as} / V_{box}$ referenced above, it is traditional to define

$$\omega_0 = \sqrt{\omega_d \omega_p}, \quad h = \omega_p / \omega_d.$$ \hspace{1cm} (20)

It follows then that

$$\omega_d = \omega_0 \sqrt{h}, \quad \omega_p = \omega_0 \sqrt{h}.$$ \hspace{1cm} (21)
The polynomials from (15) are now
\[
\mathcal{P}_1[s_] := \frac{(\alpha + 1) w_0^2}{h} + \frac{s \omega_0}{\sqrt{h} Q_{ts}} + s^2;
\]
\[
\mathcal{P}_2[s_] := \frac{\sqrt{h} s \omega_0}{Q_L} + h \omega_0^2 + s^2;
\]
\[
\mathcal{P}_3[s_] := \mathcal{P}_2[s] - s^2
\]

The denominator in each of the transfer functions \(G_d(s), G_p(s), \text{and } G_p(s)\) is

\[
\text{denom} = \text{Simplify} /@ \text{Collect}[(\text{Simplify} /@ \text{Collect}[[\mathcal{P}_1[s_] \mathcal{P}_2[s] - \alpha \omega_0^2 h^{-1} \mathcal{P}_3[s]], s] / \omega_0^4, s]^2(h Q_{ts} + h) + s^3(h Q_{ts} + h) + s^4 \omega_0^4
\]

After dividing by the constant term, this becomes

\[
\text{denom2} = \text{Simplify} /@ \text{Collect}[[\mathcal{P}_1[s_] \mathcal{P}_2[s] - \alpha \omega_0^2 h^{-1} \mathcal{P}_3[s]], s] / \omega_0^4, s]^2(h Q_{ts} + h) + s^3(h Q_{ts} + h) + s^4 \omega_0^4
\]

So the transfer functions \(G_d(s)\) and \(G_p(s)\) are

\[
G_d[s_] = \frac{\mathcal{P}_2[s]}{\omega_0^4}\text{denom2}
\]
\[
\frac{\sqrt{h} s \omega_0}{Q_L} + h \omega_0^2 + s^2
\]
\[
\omega_0^4 \left( \frac{s^2((\alpha + h^2 + 1) Q_L Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^3(h Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^4(h Q_{ts} + h)^2}{\omega_0^2} + 1 \right)
\]

\[
G_p[s_] = -\frac{\mathcal{P}_3[s]}{\omega_0^4}\text{denom2}
\]
\[
\frac{\sqrt{h} s \omega_0}{Q_L} + h \omega_0^2
\]
\[
\omega_0^4 \left( \frac{s^2((\alpha + h^2 + 1) Q_L Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^3(h Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^4(h Q_{ts} + h)^2}{\omega_0^2} + 1 \right)
\]

and the “combined” transfer function \(G_c(s)\) for the mapping \(\varphi(t) \mapsto u_d + u_p\) is

\[
G_c[s_] = \frac{(\mathcal{P}_2[s] - \mathcal{P}_3[s]) / \omega_0^4}{\text{denom2}}
\]
\[
s^2
\]
\[
\omega_0^4 \left( \frac{s^2((\alpha + h^2 + 1) Q_L Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^3(h Q_{ts} + h)}{h \omega_0^2 Q_L Q_{ts}} + \frac{s^4(h Q_{ts} + h)^2}{\omega_0^2} + 1 \right)
\]
Thus one can easily see that

\[
G_c(s) = \frac{\left(s/\omega_0\right)^2}{\left(s/\omega_0\right)^4 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right)^3 + \left(1 + \alpha + h^2\right)/h + \frac{1}{Q_s Q_s} \left(s/\omega_0\right)^2 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right) + 1}.
\]

(22)

Multiplying by \(s^2\) gives the “vented-box on-axis pressure transfer function” \(G_V(s)\) in Leach [3, §8.3], which is the transfer function for the mapping \(q(t) \rightarrow \frac{\partial^2}{\partial t^2} (u_d + u_p)\):

\[
G_V(s) = s^2 G_c(s) = \frac{\left(s/\omega_0\right)^4}{\left(s/\omega_0\right)^4 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right)^3 + \left(1 + \alpha + h^2\right)/h + \frac{1}{Q_s Q_s} \left(s/\omega_0\right)^2 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right) + 1}.
\]

(23)

In a similar manner, one can also show that

\[
s^2 G_p(s) = \frac{-\sqrt{h} \left(s/\omega_0\right)^3 - h \left(s/\omega_0\right)^2}{\left(s/\omega_0\right)^4 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right)^3 + \left(1 + \alpha + h^2\right)/h + \frac{1}{Q_s Q_s} \left(s/\omega_0\right)^2 + \left(h Q_s + Q_s\right)\left(s/\omega_0\right) + 1}.
\]

(24)

7. The response curve and the standard alignments

For plotting the normalized frequency response curve, the function of interest is

\[
|G_V(i\omega)|^2 = \left|\frac{\omega/\omega_0}{\omega/\omega_0} - \left(\frac{h Q_s + Q_s}{\sqrt{h} Q_s Q_s}\right)\omega/\omega_0^3 - \left(1 + \alpha + h^2\right)/h + \frac{1}{Q_s Q_s} \left(\omega/\omega_0\right)^2 + \left(h Q_s + Q_s\right)\left(\omega/\omega_0\right) + 1\right|^2.
\]

(26)

\[
= \left(\frac{\omega/\omega_0}{\omega/\omega_0} + 1 - \left(1 + \alpha + h^2\right)/h + \frac{1}{Q_s Q_s} \left(\omega/\omega_0\right)^2 + \left(h Q_s + Q_s\right)\left(\omega/\omega_0\right) - h Q_s + Q_s\right)^2.
\]

The coefficients of \((\omega/\omega_0)^2\), \((\omega/\omega_0)^4\), and \((\omega/\omega_0)^6\) in the denominator turn out to be

\[
c_2 = \frac{h}{Q_s^2} + \frac{1}{h Q_s^2} - \frac{2(\alpha + h^2 + 1)}{h},
\]

\[
c_4 = h^2 + 2(\alpha + 2) + \frac{(\alpha + 1)^2}{h^2} + \frac{2 \alpha}{h Q_s Q_s} + \frac{1 - 2\left(Q_s^2 + Q_s^2\right)}{Q_s^2 Q_s^2},
\]

\[
c_6 = \frac{h}{Q_s^2} + \frac{1}{h Q_s^2} - \frac{2(\alpha + h^2 + 1)}{h}.
\]

(27)
Fourth-order alignments

If there is a nonnegative number $\lambda$ such that
\[
c_2 = -4\lambda, \quad c_4 = 5\lambda^2, \quad \text{and} \quad c_6 = -2\lambda^3,
\] (28)
the response corresponds to a fourth-order Chebyshev (C4) high-pass filter if $\lambda > 0$ (see §10 Appendix A) and a fourth-order Butterworth (B4) high-pass filter if $\lambda = 0$.

The B4 “alignment” is a critical case. It requires that $h = 1$ and is possible only if $Q_{ts}$ and $Q_L$ satisfy
\[
(Q_L + Q_{ts})^8 = 8Q_L^2Q_{ts}^4((Q_L + Q_{ts})^2 - Q_L^2Q_{ts}^2).
\] (29)

Solving for each variable in terms of the other reveals that
\[
Q_L = \frac{Q_{ts}}{\sqrt{(4 + 2\sqrt{2})Q_{ts} - 1}} \quad \text{for} \quad Q_{ts} > \frac{1}{\sqrt{(4 + 2\sqrt{2})}},
\]
\[
Q_{ts} = \frac{Q_L}{\sqrt{(4 + 2\sqrt{2})Q_L - 1}} \quad \text{for} \quad Q_L > \frac{1}{2\sqrt{(2 + \sqrt{2})}}.
\]

With numbers in approximate decimal form these become
\[
Q_L = \frac{Q_{ts}}{2.613Q_{ts} - 1} \quad \text{for} \quad Q_{ts} \geq 0.383, \quad Q_{ts} = \frac{Q_L}{2.613Q_L - 1} \quad \text{for} \quad Q_L \geq 0.924.
\] (30)

So let
\[
\mathcal{M}(Q_{ts}, Q_L) := Q_{ts} - \frac{Q_L}{2.613Q_L - 1}.
\] (31)

Then a B4 alignment happens when $\mathcal{M}(Q_{ts}, Q_L) = 0$. 

\[\begin{array}{c}
\begin{array}{c}
\text{\mathcal{M}(Q_{ts}, Q_L) > 0} \\
\text{\mathcal{M}(Q_{ts}, Q_L) < 0}
\end{array}
\end{array}\]
Given a driver with known $Q_{ts}$, construction of an enclosure with $\mathcal{M}(Q_{ts}, Q_L) = 0$ is a difficult task at best. Fortunately the graph of $\mathcal{M}(Q_{ts}, Q_L) = 0$ is relatively flat over the usual range of $Q_L$ values, and hence having a precise $Q_L$ is not critical. Nevertheless, being on or close to that curve places a severe constraint on $Q_{ts}$.

When $\mathcal{M}(Q_{ts}, Q_L) \geq 0$, one normally wants a B4 or C4 alignment. For a driver with known $Q_{ts}$ our approach to this is to assume a typical value for $Q_L$ and to solve equations (28) for $\alpha$, $h$, and $\lambda$. (One can then explore the effect of varying $Q_L$.) The closer $\mathcal{M}(Q_{ts}, Q_L)$ is to 0, the smaller the resulting $\lambda$ and the closer the result is to a B4 alignment.

### Third-order alignment

If $c_4 = c_6 = 0$, the response corresponds to a third-order quasi-Butterworth (QB3) filter. This is the desired alignment when $\mathcal{M}(Q_L, Q_{ts}) < 0$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>0.0001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>B4</td>
<td></td>
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<tr>
<td>QB3</td>
<td></td>
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</tr>
</tbody>
</table>

For $\mathcal{M}(Q_{ts}, Q_L) < 0$, we seek a QB3 alignment by solving for $\alpha$ and $h$ the equations

$$c_4 = 0 \quad \text{and} \quad c_6 = 0 . \quad (32)$$

### Unification

A set of equations that encapsulates both (28) and (32) is

$$\begin{align*}
\delta (c_2 + 4 \lambda) + (1 - \delta) \lambda &= 0 \\
6 c_4 - 5 \delta \lambda^2 &= 0 \\
6 c_6 + 2 \delta \lambda^3 &= 0
\end{align*}$$

where $\delta = \begin{cases} 0, & \mathcal{M}(Q_{ts}, Q_L) < 0 \\ 1, & \mathcal{M}(Q_{ts}, Q_L) \geq 0 \end{cases}$ \quad (33)

Its solution (for $\alpha$, $h$, and $\lambda$) results in a QB3, B4, or C4 alignment as appropriate and transitions smoothly between them. Implementation in Mathematica (using a quasi-Newton method via FindRoot) shows that, with well-chosen starting values, this works well over a wide range of $Q_L$ and $Q_{ts}$ values, e.g., when

- $Q_L = 3$ and $0.1 \leq Q_{ts} \leq 0.6$,
- $Q_L = 5$ and $0.1 \leq Q_{ts} \leq 0.8$,
- $Q_L \geq 7$ and $0.1 \leq Q_{ts} \leq 0.9$.

For values of $Q_{ts}$ near the lower and upper end of these ranges, the resulting value of $\alpha = V_{as}/V_{box}$ becomes too small or too large to be practicable. Near the upper end, the C4 response also suffers from excessive ripple.
8. An example

For the HiVi M8N 8” woofer [9], the required parameters are

\[ \omega_d = 2\pi \times (33 \text{ Hz}), \quad V_{as} = 53.5 \text{ liters}, \quad Q_{ts} = 0.45 \]

We will assume that \( Q_L = 10 \) and note that \( M(0.45, 10) = 0.05 \), indicating that a C4 alignment is possible. The computed values of \( \alpha, h, \) and \( \lambda \) are

\[ \alpha \approx 0.72, \quad h \approx 0.883, \quad \lambda \approx 0.32. \]

Following is a plot of the frequency response. The dashed curves are the separate responses of the driver and port.

![Frequency Response Plot](image)

9. References


[7] math.armstrong.edu/faculty/hollis/plem


10. Appendix A: C4 alignment

When \( \mathcal{M}(Q_s, Q_t) > 0 \), one seeks coefficients \( c_2, c_4, c_6 \) so that the form of the response function \( |G_V(i\omega)|^2 \) is that of a high-pass, fourth-order Chebyshev filter [3, §8.8], i.e., so that

\[
|G_V(i\omega)|^2 = \frac{(\omega/\omega_n)^8}{\left(\frac{\omega}{\omega_n}\right)^8 + \frac{64 \varepsilon^2}{1 + \varepsilon^2} \left(\frac{1}{4}\frac{\omega}{\omega_n}\right)^6 + \frac{5}{4}\left(\frac{\omega}{\omega_n}\right)^4 - 2\left(\frac{\omega}{\omega_n}\right)^2 + 1}, \quad \text{where} \quad \omega_n = \omega_0 \left(1 + \varepsilon^2\right)^{1/8} \left(\frac{64 \varepsilon^2}{\omega_0}\right)^{1/8}. \tag{34}
\]

Here \( \varepsilon \) is the “ripple factor,” which one would like to be small, and \( \omega_n \) is the (angular) “cutoff frequency.” In terms of \( \omega_0 \), \( |G_V(i\omega)|^2 \) becomes

\[
|G_V(i\omega)|^2 = \frac{\left(\omega/\omega_0\right)^8}{1 - \frac{4 \sqrt{2 \varepsilon}}{(1 + \varepsilon^2)^{1/4}} \left(\frac{\omega}{\omega_0}\right)^2 + \frac{10 \varepsilon}{\sqrt{1 + \varepsilon^2}} \left(\frac{\omega}{\omega_0}\right)^4 - \frac{4 \varepsilon \sqrt{2 \varepsilon}}{(1 + \varepsilon^2)^{3/4}} \left(\frac{\omega}{\omega_0}\right)^6 + \left(\frac{\omega}{\omega_0}\right)^8}. \tag{35}
\]

Thus the desired coefficient values are

\[
c_2 = -\frac{4 \sqrt{2 \varepsilon}}{(1 + \varepsilon^2)^{1/4}}, \quad c_4 = \frac{10 \varepsilon}{\sqrt{1 + \varepsilon^2}}, \quad c_6 = -\frac{4 \varepsilon \sqrt{2 \varepsilon}}{(1 + \varepsilon^2)^{3/4}}. \tag{36}
\]

If we set

\[
\lambda = \frac{\sqrt{2 \varepsilon}}{(1 + \varepsilon^2)^{1/4}}, \tag{37}
\]

then the coefficients can be expressed as

\[
c_2 = -4 \lambda, \quad c_4 = 5 \lambda^2, \quad c_6 = -2 \lambda^3, \tag{38}
\]

as stated in (28). Also, solving (37) for \( \varepsilon \) yields

\[
\varepsilon = \lambda^2 \sqrt{4 - \lambda^4}. \tag{39}
\]

11. Appendix B: Computing driver parameters for a given box

With a given enclosure, one knows \( V_{\text{box}}, A_p, L_p \), and in principle \( Q_L \). These allow calculation of \( \omega_p \). The problem then is to calculate driver parameters \( \omega_s, V_{\text{as}}, \) and \( Q_t \) that result in some desired alignment. The desired alignment may be represented by a chosen value of \( \omega_s = 2\pi f_s : \text{QB3 when} \omega_s > \omega_p, Q4 \text{ when} \omega_s = \omega_p, \text{and} C4 \text{ when} \omega_s < \omega_p \). That choice will determine \( h = \omega_p / \omega_s \). Then the coefficients \( c_2, c_4, \) and \( c_6 \) in (27) contain two unknowns, \( \alpha \) and \( Q_t \), and the three equations in (33) again depend on three unknowns: \( \alpha, Q_t, \) and \( \lambda \).

The difficulty is that the system has multiple solutions (but only one relevant one), and the solution obtained by a simple root-finding method is very sensitive to the starting point. A straightforward remedy is to solve instead a constrained minimization problem,

\[
\text{minimize } \sigma (c_2 + 4\lambda)^2 + (c_4 - 5\sigma \lambda^2)^2 + (c_6 + 2\sigma \lambda^3)^2, \quad \sigma = \begin{cases} 1, & h \leq 1, \\ 0, & h \geq 1, \end{cases} \quad \lambda \in \Omega, \quad \Omega = \{(Q_t, \alpha, \lambda) | 0.1 \leq Q_t \leq 1, \alpha \geq \frac{1 - \sigma}{2}, 0 \leq \lambda \leq 0.8 \}. \tag{40}
\]

where \( \Omega = \{(Q_t, \alpha, \lambda) | 0.1 \leq Q_t \leq 1, \alpha \geq \frac{1 - \sigma}{2}, 0 \leq \lambda \leq 0.8 \} \). Implementation of this approach has resulted in the “replacement driver data calculator” that is available at [7].